

A Note On Relation Between Holographic RG Equation And Polchinski's RG Equation

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We clarify the relation between the recently formulated holographic renormalization group equation and Polchinski's exact renormalization group equation.

Jan. 2000

The holographic renormalization group flow has been clarified in the context of AdS/CFT correspondence [1] as well as in the context of open string versus closed string [2]. (For earlier attempts in this, see [3--9], also see [10].) It was noticed in these papers that the holographic RG equation arising either from the Hamilton-Jacobi theory of supergravity or from world-sheet considerations has a strong resemblance to Polchinski's exact RG equation [11].

Apparently the two equations are different. We aim in this short note to clarify the relation between them. The effective action in the AdS/CFT correspondence [12] is defined as a functional of coupling constants, while the effective action of Polchinski is a functional of the fundamental fields. Thus the holographic RG equation is naturally a differential equation in coupling constants, and the Polchinski RG equation is one in fundamental fields.

For simplicity and without loss of generality, we will consider the field theory of a single Hermitian matrix. We start with a single matrix model in 0 dimension to illustrate some of our ideas. Although in the 0-dimensional matrix model there is no infinity to remove, one still can design an artificial RG flow by introducing a “cut-off” in the quadratic term in the action

$$S_0 = -\frac{1}{2}NK(t)\text{tr}\Phi^2, \quad (1)$$

where Φ is a Hermitian matrix, $K(t)$ is the cut-off propagator, depending on $t = \ln a$, a is the running cut-off. Unlike in a genuine field theory, where once an interaction term is introduced in the action, many other terms will be generated with a nontrivial $K(t)$ in order to keep the physics invariant under changing t . In our case, there is much freedom in satisfying the RG flow, as we shall explain later. To mimic the $\mathcal{N} = 4$ super Yang Mills theory, we introduce the interaction part as a sum of single trace operators

$$S_1 = N \sum_{n \geq 3} \phi_n(t) \text{tr}\Phi^n, \quad (2)$$

where again a factor N is introduced to follow the usual large N field theory convention. With this convention, the effective action as a functional of ϕ_n defined by

$$e^{S(\phi(t), t)} = \int [d\Phi] e^{S_0 + S_1}, \quad (3)$$

has the usual genus expansion

$$S(\phi_n) = \sum N^{2-2h} F_h, \quad (4)$$

and the connected two point functions of operators $\text{tr}\Phi^n$ in the leading order is proportional to N^0 . We pause to emphasize that it is crucial to introduce only single trace operators. In the AdS/CFT correspondence, a single trace operator is related to a field in SUGRA or string theory, the role of which is played by ϕ_n in our toy model. Multi-trace operators are related to multi-particle states. The RG equation as formulated in [1] has to do with only single trace operators.

To ensure the effective action S be independent of t , S_1 must satisfy a differential equation [11]. As we shall see, this differential equation can not be satisfied by our model in which S_1 contains only single trace operators. Thus we need to relax this equation to be the one valid only when taken average in the path integral, namely

$$\langle \partial_t S_1 + K_1^{ij, lk} \frac{\partial S_1}{\partial \Phi_{ij}} \frac{\partial S_1}{\partial \Phi_{lk}} + K_2^{ij, lk} \frac{\partial^2 S_1}{\partial \Phi_{ij} \partial \Phi_{lk}} + K_3 \rangle = 0, \quad (5)$$

where

$$\langle \mathcal{O} \rangle = \frac{\int [d\Phi] \mathcal{O} e^{S_0 + S_1}}{\int [d\Phi] e^{S_0 + S_1}}.$$

we will determine K_1, K_2, K_3 momentarily. Note that the constant term K_3 can be removed by a shift of S_1 , and this shift can be absorbed into the definition for the measure of Φ . Thus in the following we will ignore this term.

For completeness, we will derive eq.(5). We start with

$$\partial_t e^S = \int [d\Phi] \left(-\frac{1}{2} \partial_t K \text{tr} \Phi^2 + \partial_t S_1 \right) e^{S_0 + S_1} = 0. \quad (6)$$

Use (5) to replace $\partial_t S_1$ in (6). Next, use the fact

$$\begin{aligned} \int [d\Phi] K_1^{ij, lk} \frac{\partial S_1}{\partial \Phi_{ij}} \frac{\partial S_1}{\partial \Phi_{lk}} e^{S_0 + S_1} &= \int [d\Phi] K_1^{ij, lk} \frac{\partial S_1}{\partial \Phi_{ij}} \left(N K \Phi_{kl} + \frac{\partial}{\partial \Phi_{lk}} \right) e^{S_0 + S_1} \\ &= \int [d\Phi] \left(N K K_1^{ij, lk} \frac{\partial S_1}{\partial \Phi_{ij}} \Phi_{kl} - K_1^{ij, lk} \frac{\partial^2 S_1}{\partial \Phi_{ij} \partial \Phi_{lk}} \right) e^{S_0 + S_1}, \end{aligned}$$

we see that if we choose $K_2^{ij, lk} = K_1^{ij, lk}$, then the second derivatives of S_1 cancel. Applying the same trick to the first term in the last line of the above equation

$$\begin{aligned} \int [d\Phi] N K K_1^{ij, lk} \Phi_{kl} \left(N K \Phi_{ji} + \frac{\partial}{\partial \Phi_{ij}} \right) e^{S_0 + S_1} \\ = \int [d\Phi] (N^2 K^2 K_1^{ij, lk} \Phi_{ji} \Phi_{kl} - N K_1^{ij, ji}) e^{S_0 + S_1}. \end{aligned}$$

Now the first term in the above can be used to cancel the first term in (6) if

$$K_1^{ij, lk} = \frac{1}{2} N^{-1} \partial_t K^{-1} \delta_{ik} \delta_{jl}. \quad (7)$$

And the inhomogeneous term is removed by choosing $K_3 = -\frac{1}{2} N^2 \partial_t \ln K$. However, as we remarked before, this term can be absorbed into a redefinition of the measure and henceforth we will ignore it. To summarize, we have derived the following equation

$$\langle \partial_t S_1 + \frac{1}{2} N^{-1} \partial_t K^{-1} \left(\frac{\partial S_1}{\partial \Phi_{ij}} \frac{\partial S_1}{\partial \Phi_{ji}} + \frac{\partial^2 S_1}{\partial \Phi_{ij} \partial \Phi_{ji}} \right) \rangle = 0. \quad (8)$$

As we advertised, this is the weak form of Polchinski's equation.

The above equation is not valid if the average symbol is removed. (We call this equation the strong form of Polchinski equation) To see this, we compute

$$\frac{\partial S_1}{\partial \Phi_{ij}} \frac{\partial S_1}{\partial \Phi_{ji}} = N^2 \sum_{n \geq 4} g_n \text{tr} \Phi^n, \quad (9)$$

where

$$g_n = \sum_l l(n+2-l) \phi_l \phi_{n+2-l}. \quad (10)$$

And

$$\frac{\partial^2 S_1}{\partial \Phi_{ij} \partial \Phi_{ji}} = \sum N G_{mn} \text{tr} \Phi^m \text{tr} \Phi^n \quad (11)$$

with

$$G_{mn} = (m+n+2) \phi_{m+n+2}. \quad (12)$$

If the original Polchinski equation applies, then $\partial_t S_1$ contains only single trace operators, and can be used to balance the single trace operators in (9). However, (11) contains double trace operators, and can not be balanced in Polchinski equation, in the large N limit, since these operators are new independent operators. In order to solve Polchinski equation, we need to introduce in S_1 double trace operators. This in turn generates triple trace operators in $\partial^2 S_1$, etc. Thus in order for the Polchinski equation to hold, all multiple trace operators must be introduced. With a little thought, it is easy to realize that this conclusion holds for any matrix model, including $\mathcal{N} = 4$ SYM. We thus learn that it is impossible satisfy the strong form of Polchinski equation without introducing multiple trace operators.

On the other hand, there is no problem to satisfy the weak form of Polchinski's equation, eq.(8). It simply generates a first order differential equations for $\phi_n(t)$. Also, as

we shall see shortly, the term $\partial^2 S_1$ in (8) is the same order as the term $\partial S_1 \partial S_1$, in the large N limit. This is quite different from the speculation of [1], where it is conjectured that the holographic RG equation is just the strong form of Polchinski equation, if so, then $\partial^2 S_1$ is suppressed by $1/N^2$.

Define the beta function

$$\beta_n(\phi) = \frac{d\phi_n}{dt}, \quad (13)$$

then

$$\langle \partial_t S_1 \rangle = \beta_n N \langle \text{tr} \Phi^n \rangle = \beta_n \frac{\partial S}{\partial \phi_n}, \quad (14)$$

where we suppressed summation over n . Use (9),

$$\langle \frac{\partial S_1}{\partial \Phi_{ij}} \frac{\partial S_1}{\partial \Phi_{ji}} \rangle = N g_n \frac{\partial S}{\partial \phi_n}. \quad (15)$$

Use (11),

$$\langle \frac{\partial^2 S_1}{\partial \Phi_{ij} \partial \Phi_{ji}} \rangle = N^{-1} G_{mn} e^{-S} \frac{\partial^2}{\partial \phi_m \partial \phi_n} e^S = N^{-1} G_{mn} \left(\frac{\partial S}{\partial \phi_m} \frac{\partial S}{\partial \phi_n} + \frac{\partial^2 S}{\partial \phi_m \partial \phi_n} \right). \quad (16)$$

Substituting (13), (15) and (16) into (8), we find

$$(\beta_n + \frac{1}{2} \partial_t K^{-1} g_n) \frac{\partial S}{\partial \phi_n} + \frac{1}{2} N^{-2} \partial_t K^{-1} G_{mn} \left(\frac{\partial S}{\partial \phi_m} \frac{\partial S}{\partial \phi_n} + \frac{\partial^2 S}{\partial \phi_m \partial \phi_n} \right) = 0. \quad (17)$$

This equation is almost the same as the holographic RG equation, say as presented in [2]. The RG equation in [2] is derived for the leading order in the large N limit. To compare with that, we use the genus expansion of (4) to derive in the leading order

$$(\beta_n + \frac{1}{2} \partial_t K^{-1} g_n) \frac{\partial F_0}{\partial \phi_n} + \frac{1}{2} \partial_t K^{-1} G_{mn} \frac{\partial F_0}{\partial \phi_m} \frac{\partial F_0}{\partial \phi_n} = 0. \quad (18)$$

Two crucial points deserve mentioning explicitly. unlike one would naively think, the term $\partial S_1 \partial S_1$ in the weak form of Polchinski equation is not identified with $\partial S \partial S$ in the holographic RG equation. Rather, it generates only the form ∂S , a correction to the beta function in (17). On the other hand the term $\partial^2 S_1$ in the weak form of Polchinski equation generates both the terms $\partial S \partial S$ and $\partial^2 S$ in the holographic RG equation, with the term $\partial^2 S$ subleading to $\partial S \partial S$ in the large N limit. Clearly, the former is identified with the disconnected part of two point functions, as also used in [2], while the latter is identified with connected part of two point functions. Clearly, both of these terms appear only in a large N theory, since they come from the double traced operators in the Polchinski

equation. With a single scalar field, one has only the ∂S term. The resulting RG equation can not be interpreted as coming from the Hamilton-Jacobi equation of a gravity theory.

In the AdS/CFT correspondence, the beta functions are determined by the local part of the effective action S . Denote this local part by S_{loc} . It contains kinetic term $\frac{1}{2}G^{mn}\partial_t\phi_m\partial_t\phi_n$, so the beta function is given by

$$\beta_m = G_{mn} \frac{\partial S_{loc}}{\partial \phi_n}, \quad (19)$$

where S_{loc} is a functional of $\phi_m(t)$ which is determined by the initial value problem. However, in our toy model there is no such a kinetic term in the extra dimension t , the reason is quite simple: the effective action (3) is defined already as a functional of initial values $\phi_m(t)$. We do not know how to define an “off-shell” action which can be expressed as an integral over the whole range of t . In fact, the beta functions are not completely determined by demanding RG invariance of the effective action only.

The 0-dimensional one-matrix model can be solved completely in the large N limit [13]. For instance, one can derive the Schwinger-Dyson equation in this limit. Apparently given a finite set of functions $\{\phi_n(t)\}$, one of them is determined by the rest by requiring RG invariance. Our above discussions serve only for the purpose of deriving the holographic RG equation from the weak Polchinski equation. Eq.(17) has a flavor of the string equation and Virasoro constraints in one-matrix model. Incidentally these equations can be derived by an action principle [14]. It may be worthwhile to pursue along this direction.

The beta-functions are completely determined in a field theory. The new ingredient here is the requirement of the cut-off independent correlation functions. It is straightforward to generalize the above consideration to one matrix model in D dimensional space-time. We use Euclidean signature. Now the regularized kinetic action is

$$S_0 = -\frac{1}{2}N \int d^D x d^D y K(x-y, t) \text{tr} \partial_i \Phi(x) \partial_i \Phi(y), \quad (20)$$

where $K(x-y, t)$ is the cut-off inverse propagator. When $t \rightarrow -\infty$, it tends to a delta function. Since we are dealing with a field theory, in order to have a RG invariant partition function, it is not enough to have a local interaction action S_1 . Assume the inverse of $K(x-y, t)$ exist (so that the kinetic term is not degenerate), denote this inverse by $K^{-1}(x-y)$:

$$\int d^D z K^{-1}(x-z) K(z-y) = \delta^D(x-y). \quad (21)$$

The weak form of Polchinski equation reads

$$\langle \partial_t S_1 + N^{-1} \int d^D x d^D y K_1(x-y) \left(\frac{\partial S_1}{\partial \Phi(x)_{ij}} \frac{\partial S_1}{\partial \Phi(y)_{ji}} + \frac{\partial^2 S_1}{\partial \Phi(x)_{ij} \partial \Phi(y)_{ji}} \right) \rangle = 0, \quad (22)$$

where

$$K_1(x-y) = -\frac{1}{2} \Delta^{-1} \int d^D x' d^D y' K^{-1}(x-x') \partial_t K(x'-y') K^{-1}(y'-y). \quad (23)$$

In order to satisfy the above equation, S_1 must contain all the nonlocal terms. If we start with a local action

$$S_1 = N \sum \int d^D x \phi_n(x) \text{tr} \Phi^n(x), \quad (24)$$

then

$$\frac{\partial S_1}{\partial \Phi(x)_{ij}} \frac{\partial S_1}{\partial \Phi(y)_{ji}} = N^2 \sum \phi_m(x) \phi_n(y) \text{tr} \Phi^m(x) \Phi^n(y), \quad (25)$$

infinitely many nonlocal terms are generated. The above can be expanded in derivatives of Φ . On the other hand,

$$\frac{\partial^2 S_1}{\partial \Phi(x)_{ij} \partial \Phi(y)_{ji}} = \sum N \delta^D(x-y) (m+n+2) \text{tr} \Phi^m(x) \text{tr} \Phi^n(y), \quad (26)$$

yielding a contact term. This contributes to the holographic RG equation a term

$$N^{-2} K_1(0) \sum (m+n+2) \int \phi_{m+n+2}(x) \left(\frac{\partial S}{\phi_m(x)} \frac{\partial S}{\partial \phi_n(x)} + \frac{\partial^2 S}{\partial \phi_m(x) \partial \phi_n(x)} \right). \quad (27)$$

This is good news, since in the holographic RG equation, this is indeed a contact term. It originates from the fact that $\phi_m(x)$ is a field in the AdS space, thus has a local quadratic term in the effective action. Denote the Fourier transform of $K(x-y)$ by $K(p)$, then $K^{-1}(p) \rightarrow 0$ when $|p| \rightarrow \infty$, and the coefficient $K_1(0)$ in (27) is given by

$$K_1(0) = \frac{1}{2} \int d^D p \frac{1}{p^2} \partial_t K^{-1}(p), \quad (28)$$

which is certainly convergent.

To have a closed form of RG equation, we thus introduce all possible single trace operators into the interaction part S_1 . A generic operator is

$$\text{tr} \partial_{i_1} \dots \partial_{i_n} \Phi \dots \partial_{j_1} \dots \partial_{j_m} \Phi.$$

Denote such a generic operator by $\mathcal{O}_I(x)$, and the corresponding coupling by $\phi^I(x)$. Now go through the above steps, we will arrive at the following holographic RG equation

$$\int d^D x \left[(\beta^I + g^I) \frac{\partial S}{\partial \phi^I(x)} + G_{IJ} \left(\frac{\partial S}{\partial \phi^I(x)} \frac{\partial S}{\partial \phi^J(x)} + \frac{\partial^2 S}{\partial \phi^I(x) \partial \phi^J(x)} \right) \right] = 0, \quad (29)$$

where all the components of the metric G_{IJ} are proportional to an integral of $p^{-2} \partial_t K^{-1}(p)$ weighted by a polynomial of p . So for each component of G_{IJ} to be well-defined, the cut-off propagator $K^{-1}(p)$ must fall off more rapidly than any negative power of p for large $|p|$. Note again that the correction to the beta function, g^I , comes from the part $\partial S_1 \partial S_1$ in the weak Polchinski equation. To compare with [2], we can identify $\beta^I + g^I$ with the beta function defined on the world-sheet, where the cut-off is defined on the world-sheet. As already pointed out in [2], there is a UV/UV relation between string world-sheet physics and spacetime physics. The two cut-offs are not identical, thus the two definitions of the beta functions are not the same. Once again, our RG equation (29) is valid for any N . Use the genus expansion (4), one recovers the RG equation of [1,2] in the large N limit. The subleading term $\partial^2 S$ in (29) is to be interpreted as coming from quantum corrections in the AdS/CFT context.

We also want to emphasize the fact that our RG equation (29) involves a single integral. This is also true for the equation derived in [2]. In order to remove this integral to obtain a local form, we need to introduce position-dependent cut-off. This is also related to general covariance in AdS/CFT. We leave a detailed discussion of this to another work.

More equations can be derived by demanding the renormalized correlation functions to be independent of the cut-off. These equations are just Callan-Symanzik equations. They can be derived from the local form of the RG equation [1], but not from (29). We can derive them in the matrix field theory by generalizing the steps leading to (29). These equations put together will give a closed system of equations for S and β^I . We are not sure whether these beta functions can be written in a form (19). Note that the metric G_{IJ} in (19) on the moduli space is the same as in (29), and is already determined in deriving (29). It would be highly nontrivial if all these beta functions are given by a single functional S_{loc} . Maybe this is the most crucial criteria for a holographic theory, and is generically violated by an arbitrary matrix field theory such as the single scalar field theory.

It remains to generalize our construction to $\mathcal{N} = 4$ super Yang-Mills theory. Although we do not see essential difficulty in doing this, we need to resolve the problem of introducing a gauge invariant cut-off. A naive cut-off will not work, since this is not compatible with

local gauge transformation which mixes all energy scales. Another way to see this is through the naive cut-off Yang-Mills action

$$\int d^D x \text{tr} F_{\mu\nu}(x) F_{\mu\nu}(y) K(x-y, t),$$

it is certainly not gauge invariant. It has been suggested to use Wilson loop as gauge invariant variable to overcome this difficulty [15], and this line of approach was followed up in [16]. However, the existence of AdS/CFT correspondence indicates that a gauge invariant cut-off exists for local gauge invariant variables. We suspect that stochastic quantization [17] may be one way to gauge invariantly regulate Yang-Mills theory. And the stochastic time has been interpreted as the RG scale recently in [18].

The discussion presented here may be viewed as a zero-slope limit of approach of [2]. However, the relation between string world-sheet and large N diagrams need to be further clarified. Also, our approach does not have the drawback of assuming perturbation theory as in [2]. The open/closed string duality need to be understood. One particularly nice example was discussed in [19].

Acknowledgments. This work was supported by a grant of NSC and by a “Hundred People Project” grant of Academia Sinica. I thank M. Yu for several useful conversations, and T. Yoneya for comments on the manuscript.

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